On Teaching Analysis, and the Role of the Calculus

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Abstract

This is a concept for teaching basic analysis at the pre-university level. It has three main characteristics:

1. We follow a *genetic approach,* tracing the history of the subject as an intrinsic motivation for the key ideas.
2. We distinguish the calculus and its *syntax* from the analysis and its *semantics.*
3. The calculus is almost trivialised by the use of technology. But we also use technology and scaffolding to ease the access to the *principal ideas and concepts of analysis.* Moreover, the tools of the calculus are complemented by some elementary methods from *numerical analysis.* The constructivist attitude is apt to mitigate the need for abstraction during a first experience with analysis.

We consider the *infinitesimal calculus an extension of elementary algebra* by rules for formal differentiation and integration. Major parts of the calculus may be automated. This fact allows to use scaffolding and the black-box – white-box principle thanks to the technology available. The purely computational tasks are reduced by the use of a computer algebra system (CAS) and numerical tools. We try to stress the understanding of notions and concepts rather than purely computational skills. The role of syntax is complemented and balanced by semantics. Technology is used in a didactical setting. Thus a CAS plays an essential role in an *experience based* approach to *key ideas.* Today's technology allows students to be faced with historical discoveries and original applications of analysis from the outset. A diligent use of technology eases the access to essential ideas, notions, experience, and insight. Moreover, we teach how to make sense of a finite machine's output – and our brains are finite too – in the context of analysis and its connections with infinities and the continuum.

All the main aspects of teaching pre-univeristy analysis are covered.

Remark: The concept outlined in the sequel took several years to evolve from a traditional calculus course into the present form by small and reversible steps. Very often the evolution was prompted by the use of technology. It boosted teaching with the joy of discovery. A growing number of constructive exercises based on historical problems began to accumulate. Many exercises designed for drill and practice in the context of former calculus instruction become obsolete when a clever use of mathematical software is part of the educational goals.

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1 On Teaching and Technology

Teaching always happens in the context of some technology. Good technology enhances teaching, while bad technology creates a need for more instruction to overcome the shortcommings of the tools available.

Today's technology demonstrates that large parts of the calculus may be automated and performed even with the help of handheld devices. This fact challenges educators and teachers of mathematics alike. *Is the calculus still a worthy subject in the context of general education? Should our syllabus be adapted to this situation? And if so, what might have to be changed or rearranged in a novel way?*

This text reflects on the evolution of mathematics from early geometry to classical mathematics with special attention to the co-evolution of the tools available from compass and straight edge to paper and pencil, from logarithms and slide rules to calculators, work stations, smart phones or tablets. This point of view clearly suggests a genetic approach to teaching mathematics. In particular, I propose to study *some key problems that lead the way from Greek mathematics and technology to classical mathematics and present day technology.* Doing so, we focus on the genesis of algebra, the calculus and analysis. The insight gained from this reflection is the main reason for our approach to teaching the calculus and the principles of analysis using modern tools.

It will become clear that the use of technology may naturally entail a reduction of some skills our predecessors were proud of. The formation of basic notions essential to analysis (i.e. semantics) may be separated from the rather syntax oriented calculus. Indeed, calculus is an extension of basic algebra by the formal rules governing the differentiation operators (i.e. syntax). The *understanding of basic notions* cannot be replaced by computational skills. Understanding the basics is necessary if we aim to teach relevant applications, e.g., applications based on *dynamical systems.* Today's tools and a sound conceptual basis allow to introduce both kinds of dynamical systems, discrete or continuous. The discrete case is essential in many numerical procedures or in evolution models with discrete time steps. The continuous case provides the stage for models based on ODEs. The two cases are linked with one another by discretisation or the reverse process where a limit of a sequence of discrete processes ends up as a continuous one. The *interplay between discrete and continuous structures is a leitmotif in teaching elementary analysis.* Algebra and algorithms, and hence any CAS, are bound to work by finite means. But they may operate symbolically on items from the mathematics of continua, in particular symbols for real numbers, and continuous or differentiable functions, vector fields, to mention but a few.

The following approach favors the *genetic method* over an imitation of Bourbaki style mathematics. The formalisation of mathematical content is a necessary task too. However, history shows that formalisation rather follows than preceeds experimentation, discovery, exploration and accumulation of insights and results.

An alternative approach used in higher education stresses the logical structures by definitions, theorems and proofs. This style of teaching is time saving and ends up with a lean formal theory whose meaning or possible interpretations remain largely untouched. If applied to unprepared beginners it risks to miss the point and to produce formalisms of little use.

Euclid codified Greek Mathematics up to his time in the Elements only after a long and prosperous evolution of geometry, logic, number theory. In his lifetime the decline of Greek power became obvious. This may indicate that formalisation also serves to concentrate and reduce a wealth of related but scattered ideas in an axiomatic manner. In this way the 13 books of Euklid succeeded to preserve the legacy of Greek Mathematics. At the same time

Euclid set a standard for teaching geometry in a canonical and deductive way. It persisted largely untouched in Europe and elsewhere during 2 millennia. Thus Euclid became a role model for mathematicians of the formalist school. In 1899, Hilbert, a leading formalist, published his *Grundlagen der Geometrie*, a novel systematic analysis of axiomatic geometry. The discovery of Non-Euclidian geometry in the 19th century may be taken as a major motivation for this advance.

Remark At school, a formal definition of the *real numbers* is avoided, while a naive view on the real line is adopted. A competent formal treatment of the reals is left to courses at the univeristy level.

However, the extension of numbers $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{R} \subset \mathbb{C}$ may be suitably explained at school in the context of algebra and hence by finite means.

Much analysis and calculus was in use for more than a century, before an attempt was made to understand the completion of the rationals $\mathbb{Q} \subset \mathbb{R}$. This step was possible only in the 19th century with major contributions by Cauchy and Dedekind. It is the essential step in creating a *continuum* all of whose properties are necessary to prove the major theorems of elementary *analysis.* The syntax of the calculus is a much simpler affair and is valid without this essential step. The *existence* of limits, derivatives, and primitive functions must be granted by analysis. We also add that the complex numbers were defined algebraically before the completeness of the reals was dealt with in analysis.

Similar remarks apply to the notion of function. Newton took it for granted, that any function may be represented locally by a power series. The concept of function used in contemporary analysis is rooted in set theory that was available only towards 1900. Today, Newton's concept would be described as a special type of function, now called 'analytical'.

2 The Origins of Analysis and the Calculus

Greek geometers used to think in terms of ideal tools, ruler and compass. A good part of geometry was constructive, even *algorithmic* in nature and hence *finitistic* by ideology. Constructive solutions to geometric problems were restricted to finitely many fundamental constructions with ruler and compass. Another typical expression of Greek mathematics is found in logics and the need for proofs. All proofs justifying certain constructions or the derivation of theorems from fundamental principles were finite chains of arguments based on a finite set of axioms or definitions. By the way, the Greek atomists' theory of matter follows strictly analogous principles.

Moreover, *algebra* was unavailable in Greek antiquity. Only much later, algebraic methods helped to ease the presentation of theorems and their proofs also in geometry.

2.1 Three Famous Problems Greek Geometers Tried to Solve in vain

We might compare ruler and compass with the Turing machines. Then we discover a close analogy between classical problems in geometry and recent problems in algorithmic and computational mathematics. Greek mathematics left us with three famous unsolved problems:

- Given the side length of a cube. Construct the side length of the cube whose volume is twice the volume of the given cube using ruler and compass only.

- Given an arbitrary angle. How can the angle be trisected by using ruler and compass?

- Given an arbitrary circle. How can the area of the circle be represented by a square of equal area by using ruler and compass only?

None of the three problems may be solved using the prescribed tools. But this insight needed 19th century mathematics for a proof, mainly radically new ideas in algebra. Note that the trisection of angles as well as the solution of cubic equations would be possible by Origami constructions.

2.2 Infinity, Lacking Concepts, and Paradoxa

Infinity transcends human imagination. Greek philosophers were aware of the problems arising when the firm ground of finitistic methods is left behind. Still some may have invented certain paradoxa of the infinite in order to demonstrate the pitfalls of naive thinking combined with talking about infinity. Zenon's paradoxa [Achilles and the tortoise, the dichotomy and the parodoxon of the flying arrow] may serve to signal the dangers of naive thinking or the lack of well-defined technical terms still today. We possibly follow Zenon's intentions by recalling and commenting them before entering the territory of analysis/calculus.

One possibility to avoid the paradoxa is to follow the atomists whose world view is essentially finitistic and discrete. Their mathematics would reduce to combinatorics. Possibly Archimedes probed this perspective when he wrote *The Sand Reckoner*.

An alternative needs definitions and notions that avoid the paradoxa. The paradoxon of Achilles and the tortoise as well as the paradoxon of the flying arrow come down to the insight that a scientific definition of velocity is connected to concepts of space and time which have to precede such discussions. The problem of the dichotomy has to be overcome before numerical methods such as bisection may be understood.

The good news is that Archimedes found a way to overcome the problems hidden in Zenon's paradoxa. This is the main point of the next section. Archimedes' way of thinking paves the way for introducing all major concepts of basic analysis in pre-college mathematics.

The notion of velocity emerged only in medieval times (Oresme) but remained enclosed in academic circles. A breakthrough happend with Galileo's and finally with Newton's work in physics.

2.3 Archimedes: Engineer, Mathematician, Genius – Early Roots and Principles of Basic Analysis.

Ruler and compass constructions allow approximate solutions up to any given positive tolerance. But this could not satisfy the requirements of rigorous philosophical standards in classical Greek mathematics.

Think of an engineer designing a gear box or a mechanical orrery. In its construction wheels will have to be made with mutually prescribed proportions of the lengths of their circumferences. Greek mathematicians knew that squaring the circle and measuring the circumference of any circle are essentially the same task. A mechanical orrery of surprising sophistication made in ancient time was found near the island of Antikytera around 1900. Hence, antique craftsmen or engineers knew how to use approximations to squaring the circle of prescribed precision with success.

For the sake of preparing the essential ideas of analysis allow me to personalise a major step of mathematical thinking. The following sketch is a kind of saga with a grain of truth rather than historical truth with a grain of salt.

About Teaching In class, I might read now Cicero's 4th letter from Tusculum where he searched and found Archimedes's tomb stone and found engraved a cone, a cylinder and a sphere. Then I would discuss Archimedes's *approximations* for π which of course are the essential approximations for the area of the unit circle by regular n-gons, inscribed and circumscribed. Today this provides an exercise in programming, and it is instructive because it may show the pitfall of naive algorithms and the risk of numerical degenerations due to the use of machine numbers instead of real numbers. Here we are close to a major issue of analysis and we dive into the problem rather than to avoid getting wet by constructing the real numbers in a 19th century fashion outside the intellectual reach of youngsters. And we may choose this approach only thanks to the easy acces to a programmable device be it a handheld, a tablet or a desktop.

The main point here is that Archimedes thought like an engineer. He overcame an ideological obstacle of Greek geometry by accepting a method that lead to approximate solutions with an error that could be reduced beyond any given positive tolerance. This procedure is good enough for all practical purposes. It marks a change in attitude and is an essential step necessary for a procedure to be referred to as the *Archimedian trick.* It is an *essential ingredient for basic analysis.* Moreover, it paves the way to the more sophisticated modern limit concept.

Once we understand how to approximate the area of a unit circle up to a given arbitrary positive tolerance, we may satisfy the requirements of an engineer or craftsman who has the task to construct a mechanical device within the limitations given by his materials and his tools. In this way we are able to measure the content of any cylinder, cone or sphere for the sake of practical applications. In doing so, we use *discretisations* of an appropriate kind in order to produce results that can be proved to satisfy the precision requirements. Technically speaking we consider Riemann sums approaching the content of a circle or half circle, a cone, a sphere, and we may do so because the algebra involved gets trivialised by *scaffolding* thanks to the CAS which simplifies the sums where necessary. Note the advantage of using algebra, a method not available to Archimedes.

Archimedes might have wondered whether all plane domains with curved boundaries might be as hard to square by ruler and compass as the circle seemed to be. He studied the case of a parabolic segment and found a way to construct a square of equal content by a classical construction with ruler and compass. This example may be imitated in school with the help of a CAS. An immediate continuation of his line of thought is to be found in Kepler's work on the volume of barrels. Kepler's rule or its sibling, Simpson's rule, are early examples of numerical methods for integration. Both may be used as meaningful examples in productive exercises and elementary applications of analysis.

Furthermore, a discussion of an ancient atomist's view may be fruitful here. For an atomist, the iterate refinements stop after a *finite* number of steps. In his view the area or volume reduces to a *finite* sum over the corresponding measures at the atomic level. Measuring is reduced to counting. (cf. also Archimedes's writing on the sand reckoner)

The term *Archimedes's trick* is a label to denote the method of successive approximations up to an arbitrary positive tolerance. And, moreover, the radically new convention is that a problem is considered to be *solved* (not only for all practical puroses) if this stage is reached. This suffices, of course, for all constructive approximations, and we may stay inside the rational numbers as long as we refrain from postulating the existence of limits.

The so called Archimedian property postulates that given any two positive numbers $0 < a < b$ there exists a natural number n such that $n \cdot a > b$. This axiom rules out the existence of positive numbers all of whose multiples by natural numbers are smaller than 1. Leibniz thought that positive infinitesimals would have just this property.

In all the examples studied, the limits may be determined naively by algebraically grouping terms into a constant and terms involving the discretisation parameter. As the case may be, this parameter is n and we consider $n \to \infty$ or it is some Δx and $\Delta x \to 0$, and in the examples chosen the error terms are functions of the discretisation parameter and these functions tend to zero if the discretisation is pushed to its continuous limit.

Archimedes's trick is now applied in the educational context to explore the diverse methods for approximating the volume of a sphere (stacks of thin discs, triangulation, shelling) in analogy to familiar methods for approximating the area and the circumference of a circle. By way of example, we are lead to consider difference quotients for the area of a circle, as well as the volume of a sphere, as functions of the radius. In this we get into contact with the *idea* of the derivative and of the integral. But beware – the name is kept in the cheek as well as 'integral' is a no-name at this stage of instruction. It is noteworthy to observe that both examples might serve to illustrate even the role of the Fundamental Theorem of analysis, yet we cover this fact with silence. But we might keep it in mind in order to show – in the context of matured ideas and clear concepts – how the Fundamental Theorem establishes a link between integrals and derivatives.

Archimedes's intuition found some basic principles and essential *ideas* on the way to analysis. For lack of algebra he was unable to contribute to what we now would call the calculus.

2.4 Heron and Iterative Numerical Approximations to Roots

Square roots may be constructed by ruler and compass respecting the highest standards of Greek philosophy. But this does not help to satsify practical needs for a good approximation in terms of fractions. Archimedes's trick does not work with the materialised tools of geometry.

The long tradition of mathematical culture in the Middle East offered an alternative. Its origin is documented in clay tablets but the name of the method is attributed to Heron, a physicist, engineer and mathematician living in Alexandria in the Hellenistic period.

Heron's method is interesting because it shows the role of *numerics* in antiquity. It produces a sequence of rapidly converging rational approximations to square roots by iteration. This may be exploited to extend the role of Archimedes's trick beyond geometry to function iteration producing successive approximations to solutions that cannot be reached in finitely many steps within the given arithmetic.

About Teaching In practical computation with calculators, saturation of the decimal representation occurs. Still an algebraic argument shows that errors decay in exact arithmetic. Hence Archimedes's trick applies and shows the existence of an exact square root to any positive number independently of geometry and its idealised tools.

Students' projects on how to adapt Heron's method to arbitrary roots are an option here. They may lead to Newton's famous method being rediscovered in the context of special examples in school.

Heron in a modern perspective: Heron's iteration is a discrete dynamical system on the field of rationals $\mathbb{Q}$. Typically its fixed points lie in $\mathbb{R}$ or $\mathbb{C}$. This fact motivates the need for extensions of $\mathbb{Q}$ in the context of a smooth mathematical theory. But paradoxically all numerical computations take place in some finite subset $\mathbb{R}_{\mathbb{M}} \subset \mathbb{Q}$. Even a CAS is bound to a finite memory implying that its range of effectively computable terms remains finite all the time. Our tools do not fit the needs of the basic mathematical background in theoretical analysis. However, the calculus provides us with some algebraic shortcuts which open the way for solutions in finite terms under suitable conditions. The impression of an allmighty calculus is seductive like a mirage. Numerical approximations are absolutely essential to the engineers' tricks of trade. Hence *numerical analysis is a must in general education* complementing the applications of the calculus in an essential way.

2.5 Dido, Pappus and Fermat: Is Nature Ruled by Extremal Principles?

Tell about Dido and the foundation of Carthago, tell about Pappus and his school. In particular tell how Pappus explained the propagation of light rays between two points A and B and why they ought to follow a straight line or why the angle between the incident light ray and a mirror is equal in size to the one of the outgoing light ray because of the postulated parsimony of nature.

An easy argument in Euclidian geometry suffices in order to 'prove' what Pappus claimed. But it only works under the hypothesis that shortest paths between two points are straight lines.

Tell that Pappus also claimed the bee hive to solve a minimal problem. Another claim of his says that the bee cells have hexagonal walls 'because' the bees have six legs.

An exact study reveals that bees aren't precision workers and that several different types of bee cells may be found in nature. Still T. C. Hales in 1999 published a proof that a form of the honeycomb conjecture attributed to Pappus or Varro is true in two dimensions.

Twelve centuries after Pappus, Fermat extended the principle of the shortest path for light rays and gave it a new twist. Years before O. Rømer was able to establish the finiteness of the speed of light, Fermat claimed that light propagates at finite speed depending on the medium of propagation. His principle was little more than a belief that light propagates through an optical system such as a rain drop, a lens, a prism, . . . in such a way that the time of travel along a light ray was minimal [or extremal] among all paths connecting a given starting point A to some given point B.

About Teaching We take up Fermat's claim and derive the *law of Snell* in mathematics as an exercise and application of Fermat's principle. [Here we need an essential ingredient to modern analysis: analytic geometry and its link to algebra.] Moreover we enlarge the set of exercises in nonlinear optimisation. They may be solved first by using the CAS and *scaffolding*. Later on recall the *black box – white box principle* to make acquaintance with numerical optimisation and with Fermat's forerunner for the derivative. In the case of a polynomial function p, the difference quotient $\frac{1}{h}(p(x+h) - p(x))$ may be divided out completely and then we recognize a function of x and a multiple of h. Setting $h := 0$, the function of x remains and it is what we now call the *derivative*. It is noteworthy that Fermat didn't hit the correct definition at first stroke. We use the 'error' of the famous man for didactical purposes and comment on it later on when the derivative will be defined formally.

This example is relevant since in the 20th century all physical theories were based on variational principles. This view allowed to see parallels between optics and mechanics and to free a way for quantum mechanics to amalgamate both subjects.

2.6 Kepler and the Volume of Barrels

When Kepler married his second wife, the wine was furnished in several barrels. Kepler wundered how the seller of the wine determined the content of the barrels. The volume given for the barrels was non-plausible to Kepler. This prompted the scientist to derive a handy formula to determine the volume of a barrel based on four measurements: The cross sections of the barrel at the bottom, at the top and at mid height, and the distance between bottom and top. *Kepler's rule* is a formula that approximates the volume of a barrel based on these data. The formula is exact, e.g., for cylinders, truncated cones, spheres and their segments or spherical zones, and in general it provides good approximations for many practical applications.

About Teaching Kepler's method may be rediscovered in school. A CAS calculator and scaffolding allow to concentrate on the essential ideas:

- interpolate the measures of the three *cross sections* quadratically.
- integrate a quadratic function over an interval using discrete approximations and idealisation by limits.

It is worthwile to extend *Kepler's rule* and to discover *Simpson's rule.* It provides us with an opportunity for an exercise in programming an early example of a numerical procedure. This leads to first experiments in what we shall call *numerical integration* only after a systematic categorisation of all the diverse examples to be given in the sequel. Numerical integration nicely complements formal integration whenever the primitive function of some elementary function misses to be elementary itself.

By the way, Kepler was among the first ones to profit from the then novel use of logarithms. Without the tables, his lifetime might not have sufficed to decypher the laws of planetary motion based on the records of the best astronomical observer of his time and his predecessor at the court of Emperor and King Rudolph II in Prague, Tycho Brahe.

2.7 Aristotle, the Peripatetics, and Galileo

Aristotle was an authority and part of his prestige derived from the fact that he was the teacher of Alexander the Great. Artistotle's philosophy embraced also an early form of natural science which dominated what was taught, learned and believed in European schools and universities well through the Middle Age. Aristotle's followers, the peripatetics, claimed that a falling object accelerates in such a way that the speed is proportional to the length of the path coverd. Galileo contested this claim and was lead to think and experiment with falling bodies. His empirical approach to physics helped overcome the imposed mistaken beliefs and stagnant understanding of the natural phenomena under the authoritative regime of traditional philosophy as well as religion. Galileo dealt with models of motion under constant acceleration and hence was able to describe the trajectory of arrows or mortar shells much better than the learned philosophers or the gunners. Galileo placed mathematics at the base of understanding nature by saying that the book of Nature lays open before our eyes but it is written in geometric figures and who wants to read has to master the *language of mathematics.*

About Teaching Essentially Galileo defined mean velocity as a difference quotient of positions and [mean] acceleration as a difference quotient of velocities. The case of Galileo also shows that notions are basic to understanding rather than formulae.

2.8 Pascal and Leibniz: Infinitesimals and the Calculus

Pascal, a child prodigy, found about 400 theorems about projective geometry and conic sections. Among other results he showed how to construct a tangent to a conic section $\mathcal{C}$ defined by five points through one of the given points. Call it $P \in \mathcal{C}$ and *imagine* the results of the following procedure. Step 1: Select an auxiliary point $Q \in \mathcal{C}$ and construct a secant line through P and Q. Step 2: move point $Q \in \mathcal{C}$ towards P and observe what happens to the secant line. In this perspective, one may catch the seductive idea that the tangent to $\mathcal{C}$ in point P is a line connecting P with itself as the limiting case of a secant between 'immediately neighboring' points $P \approx Q$ both on $\mathcal{C}$. The idea that tangents are limiting cases of secants must have been somehow in the air. Or not?

It is noteworthy that Pascal constructed one of the early mechanical calculators, cf. [PAS]. It is also known that Leibniz visited Paris and must have been shown Pascal's calculator, *la pascaline*. Leibniz was impressed by Pascal's work. As a matter of fact, Leibniz tried to surpass Pascal and set out to invent a calculator of his own.

His differential calculus opened a way to early differential geometry and allowed to compute tangents for almost any curve known at his time by using the *somewhat mysterious differentials.* Leibniz's differentials were conceived as *positive but infinitely small.* His concept is suggestive but cannot be grasped easily. If 'number' means fraction there is no place for infinitesimals. Infinitesimals contradict the so called archimedian property of real numbers. A well established notion of real number was lacking at the time. The geometric substitute, a straight line, was diffuse enough to allow speculations about the local structure of a continuum. Leibniz successfully developed a calculus for differentials based on an extension of algebra to include operations with differentials and a pertaining convincing notation. Leibniz was focussed on syntax. His calculus was rather suggestive and allowed to find correct answers to many questions essentially by following the rules of a syntax without an obvious semantic basis. Leibniz was successful and competitive with the leading mathematicians of his time. Although he was a lawyer and librarian by profession, his creative mind made him a universal genius. His character brought him into conflict with others of similar stature.

About Teaching In our introduction, we paraphrase the computation of tangents in guise of *local linear approximations* following the two steps of the archimedian trick. The slope of the tangent to the graph of a function f is gotten by first computing the slope of a secant.

$$DQ(f, x, \Delta x) := \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

In the second step, we try to idealize and let $\Delta x \to 0$. Of course we discuss the problem of existence and we find that in the case of polynomials (of low degree), algebraic manipulations suffice to find the limit. By the way, this is Fermat's view on the problem.

Our first aim is to be able to find derivatives for all polynomials. This is achieved by proving two results of the calculus: The rules for derivatives of sums and of products. Calculus now suggests as corollaries the rules for integral powers and even fractional powers.

Leibniz propagated the idea that thinking could be formalised and hence mechanized with drastic consequences: Stop quarrelling – Let's compute!

It is true that the infinitesimal calculus was crucial in helping the early practionners to advance by purely formal computation according to rules of the calculus rather than by insight into the semantical framework of the new methods. A quote by Jean d'Alembert fits neatly into this context: ≪*Allez en avant et la foi vous viendra !* ≫.

Insight beyond computation or symbol manipulation was acquired during the 19th century when the foundations of analysis were improved. Only then such basic objects as *real numbers* or *limits* or *continuous functions* were defined formally.

An instructor may reasonably expect students to solve certain problems by applying the calculus and its rules before they master the basic concepts of analysis. This may be seen as progress due to lucky circumstances and clever teaching methods. Still, when teaching is focussed only on the calculus, the sight narrows. This may lead to troubling illusions. Any CAS can do the calculus but it cannot understand the concepts of analysis.

The purpose of this note is to convey a balanced view on semantics and syntax. Thus, it is only fair to exhibit the advantages and drawbacks of both. A strong syntax has its advantages because it may reduce the solution of problems to algorithms and computation or symbol manipulation. However, without a semantic backup by analysis, the calculus alone risks to remain mysterious or to lead astray.

To some extent, a CAS makes Leibniz's vision to be true. But Daniel Richardson in his 1968 thesis put an end to Leibniz's dream. Some essential questions are algorithmically undecidable in the class of elementary functions a CAS is supposed to operate on, cf. [RIC].

2.9 Newton's Principia: New Mathematics for New Physics

While Kepler found his laws of planetary motion by analysing empirical data, Newton made a claim about the law of gravitational attraction. However, a problem remained, the attraction exerted by masses escaped laboratory experiments in Newton's generation. The celestial bodies with their huge masses might be used to corroborate the claim. Newton's idea was to deduce Keplers laws of planetary motion as a logical consequence under the hypothesis of his universal law of gravitation. This is what Newton finally achieved. Still he had to invent new mathematical tools in order to finish the proof. Finally, Newton invented his own version of derivatives and integrals and he based mechanics on axioms and differential equations. With respect to his masterpiece, the *Philosophiae Naturalis Principia Mathematica* containing the solution of the *two body problem*, he was modest enough to remark 'I stood on the shoulder's of giants', acknowledging thus the role of many forerunners in a long tradition.

Newton's presentation however eschewed this new form of mathematics and rather relied on geometric arguments. (Was this a 'didactical' concession to his contemporaries or rather a protection for the uniqueness of his powerful invention?) Although he introduced the new discipline of *dynamical systems* with his celestial mechanics, we recognise this fact only with hindsight.

Finally Newton and Leibniz disputed bitterly over the priority of the invention of the calculus.

About Teaching While teaching analysis to beginners, we profit from the occasion and introduce some connections between mechanics of mass points and analysis. We consider a mass point moving along the real line. Its *position* x is a function of time $x : t \mapsto x(t)$. From any given position function we may tentatively derive a *velocity* function in two steps.

- Choose an interval $\Delta t \neq 0$ and define the mean velocity for the interval $[t, t + \Delta t]$ by the difference quotient

$$\bar{v}(t, \Delta t) := \frac{1}{\Delta t} \cdot (x(t + \Delta t) - x(t))$$

- Remove a possible discretisation error by taking the limit $\Delta t \to 0$ and define

$$v(t) := \lim_{\Delta t \to 0} \bar{v}(t, \Delta t) \quad \text{if this limit exists}$$

Remarks

1. Note that here the derivative naturally appears as an *operator* producing functions from functions, rather than a prescription to produce numbers. This is essential for the next step where acceleration is derived from velocity by an analogous procedure.

2. Acceleration is the derivative of the velocity function:

$$a(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot (v(t + \Delta t) - v(t)), \text{ if this limit exists}$$

3. Standard notations

   (a) Newton: $v = \dot{x}, \quad a = \dot{v} = \ddot{x}$
   Today, this notation is reserved for derivatives if the variable is time t.

   (b) Leibniz: $v = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

   (c) Cauchy: For all t in the domain of the functions:
   $v(t) = x'(t), \quad a(t) = v'(t) = x''(t)$

4. Newton's law:
   $F = m \cdot a = m \cdot \ddot{x}$ is a *differential equation* for the function $x : t \mapsto x(t)$.

5. Galieo's inertial law has the form $m \cdot \ddot{x} = 0$. This implies that either $m = 0$ or $\ddot{x} = 0$. In the second case, Galileo claims this to be equivalent to $v(t) = v_0$, a constant.

   A mathematical proof of this claim requires two arguments. The derivative of a constant v_0 vanishes as any beginner who understood the concepts will agree. The opposite conclusion, however, requires all the major theorems of elementary analysis, possibly disguised in the Fundamental Theorem of analysis.

6. Any [small] mass falling [close to the surface of Earth and without air resistance] obeys a law of the form $\ddot{x} = g = \text{constant}$ according to Galileo.
   We may check as an exercise that $\frac{\mathrm{d}}{\mathrm{d}t} g \cdot t = g$ and that $\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \cdot gt^2 = g \cdot t$, but for all constants x_0 and v_0 the functions $x : t \mapsto \frac{1}{2} \cdot g \cdot t^2 + v_0 \cdot t + x_0$ are solutions too. How can we see that no other solutions exist? Could mathematics possibly produce solutions that Galileo missed or, even worse, that nature forgot to display?

   Those who venture considering the vector version of this example will find the parametrisation for a parabolic flight under a constant acceleration.

Final Remark The historical prelude to analysis comprises a period of roughly 2 millennia in the history of human thinking. This is the epochal duration during which major figures contributed ideas, notions, results in their quest for new mathematical *understanding*. It seems fair and wise to devote a substantial amount of time while teaching precalculus by mimicking the evolution of essential ideas towards analysis. Studying relevant contributions to the genesis of early analysis in historical and biografical perspective may open the mind for curiosity and even motivate students to acquire formal mathematical skills. A systematisation and formalisation will follow later on.

3 Elementary Analysis: Approximate to Exact or Discrete to Continuous – Back and Forth

The diversity of the historically documented examples is likely to favor the wish for simplification and clarity. Now comes the chance to categorise the examples and to exhibit the basic principle anew. *Archimedes's trick is used to bridge the gap between the discrete mathematics and the mathematics of the continuum.* We only now define the key notions of *derivative* and *Riemann integral* and demonstrate their ability to simplify and structure the set of examples. Various applications of the derivative or the integral will be identified in the examples. The historical examples allow a preliminary limitation to work with polynomial functions. This is a didactically desirable simplification as essential as the use of the black-box – white-box principle with the help of a CAS.

Rational functions, exponentials and logarithms, trigonometric functions and the cyclometric functions will be discussed later on. For each class of functions a special aspect will be dealt with. Analysis is by no means reduced to plotting graphs and computing extrema, tangents or areas. With the help of a CAS we are able to reach such subjects as approximations, polynomial approximations and series expansion, numerical methods that may turn the numerical solvers or the numerical integrations of the CAS into white-boxes. Modelling is the root for extremal problems as well as for elementary ODEs or dynamical systems. The CAS allows to make the students fit for understanding why analysis is essential in the context of many applications in the sciences or in engineering. It will also be clear, why in many modern applications analysis does need the support from numerical methods, algorithmics and computer science.

3.1 Discretisation

If a problem involving some continuum and hence infinitely many operations it may be reduced to a finite one by discretising. Discretisation creates some errors and loss of information. The problem is to keep discretisation errors under control. If this can be done sucessfully, a second step will follow and purge the effects of discretisation. (cf. Idealisation, below)

Examples

- *Mean values* of finitely many values sampled from some continuous function.
- *Riemann sums* for a function sampled at finitely many points.
- *Difference quotients* as the mean specific rate of change over an interval.
- *Function plotting* is always based on finitely many samples.
- *Difference equations* e.g., Euler's method, sampling a vector field

3.2 Idealisation

The errors of discretisation may often be removed by taking a suitable *limit.* Here some general properties of functions like continuity or differentiability or smoothness are usually required.

Basic analysis defines and limits its range of validity to such cases where the two principles of discretisation and idealisation may be combined sucessfully. Counter examples may serve as a warning. However their mathematical treatment easily exceeds the possibility of a first course. Now the advice of d'Alembert cited on page 9 becomes a wise didactical decision.

4 Key Notions, Examples, and Comments

On Machine Numbers, Rational Numbers, and Real Numbers Every student working with a computer, even a handheld calculator, ought to be confronted with some of the unavoidable difficulties of numerical computing: roundoff, underflow, overflow as consequences of the finiteness of the set of machine numbers. Similarly, despite the famous pythagorean dogma concerning the non-existence of irrationals, the rational numbers do not suffice and the real numbers, which are needed for analysis to work, typically have no names and pop up like random numbers from an uncountable set.

The Derivative exists if the limit of the differential quotients exists. The term 'derivative' has several interpetations.

- value of a function
- a function
- a linear operator on function spaces.

The calculus refers mainly to the rules that hold for computing the derivative of some elementary functions based on a few generic rules and some finite set of rules for special analytic functions.

The Integral comes in various forms:

- the limit of Riemann sums defines the *Riemann integral* or *definite integral.*
- an *integral function* is defined as $I_f : x \mapsto \int_a^x f(t)\mathrm{d}t$, where each function value is given by a Riemann integral.
- an *indefinite integral* for a given continuous function f is the solution set of the *differential equation* $F' = f$.
- for a given continuous function f, a *primitive function* F satisfies $F' = f$ and hence is an element of the indefinite integral.

All these notions are connected by the statement of the Fundamental Theorem of analysis. The definite integral of a continuous function may be defined as a limit of Riemann sums. The indefinite integral is the solution of a differential equation and the Fundamental Theorem assures the existence of primitive functions with continuous derivatives. Moreover it describes how to get primitive functions based on definite integrals and it parametrises the indefinite integral, i.e. all the solutions of the differential equation $F' = f$. The proof of this central result is based on all major theorems of elementary analysis and cannot be done by the means of calculus. Indeed, the calculus cannot assure the existence of a primitive function for any continuous function $f : I \to \mathbb{R}$ on a compact Interval I as is done by the Fundamental Theorem of analysis.

Remark In the perspective of the calculus the Fundamental Theorem offers a way to evaluate definite integrals via primitive functions. Although primitive functions exist abstractly for all continuous functions given on some compact interval, it is known that some primitive functions of elementary functions are not elementary. Here the power of the calculus – and hence of any CAS – gets limited. Effective numerical approximations are unavoidable.

Often the exact evaluation of functions is taken for granted, especially in the context of exact formal solutions. This expectation often conflicts with reality. *Numerical analysis* shows how to construct *sufficiently precise and computable approximate solutions to many problems of practical significance,* among others the efficient approximations to standard functions like, e.g., logarithms, the trigonometric functions, and their inverses.

Thanks to technology, automated *numerical integration* becomes an alternative to formal methods. However this alternative typically involves sampling and discretization and some errors that cannot be removed at will by effectively computing a limit.

Sampling and [spline] Interpolation The use of graphing devices calls for some remarks about sampling. Sampling is necessary for computer graphics. It may cause artefacts that illustrate the problems of discretisation in this case. Similarly function tables involve sampling too. The reconstruction of approximate values for a function given by a list of sampled values is mostly done by spline interpolation. Interpolation is an alternative for closing gaps *independent* of the concept of limits.

Mean Values and the Fundamental Theorem of Analysis The mean value of a continuous function over a finite and closed interval may be computed either by a definite integral or by a difference quotient (involving a primitive function). Both methods lead to the same answer. This is the *essence* of the Fundamental Theorem of analysis.

Differential Equations make the essential difference between mathematics rooted in antiquity and emancipated thinking during the enlightenment. Archimedes already addressed problems such as computing arclength, areas, volumes, barycenters, Pappus thought about how extremal priciples might rule the world. But only Newton's genius opened a new road to what we now call dynamical systems. This decisive step has to be addressed in school.

In view of the role ODEs played in the genesis of analysis, some differential equations ought to belong to any introduction to analysis. Hereby the graphing capability and *numerical simulation* are essential. Moreover the intuitive idea that the motion of a liquid may be described by either a family of streamlines or by a vectorfield describing the velocity of the particles at any place links two key concepts: The vectorfield is the geometric image of the ODE and the streamlines are parametrisations for the paths of the moving particles. This simple physical idea might be more illuminating to a general audience than the bag of tricks offered by the calculus when formally solving some special ODEs. Any such trick is of limited use only, while the key idea of streamlines and the pertaining vectorfields is of general use. We pass from streamlines to the velocity vectors by differentiation and the way back by integration.

At the Limits of Basic Analysis All cases where discretisation combined with idealisation by limits fail, are excluded from basic analysis by definition.

Fractals might be dealt with in school although many questions beyond the reach of the tools of basic analysis might pop up. Here too computers opened new perspectives for curiosity driven mathematical investigations by students.

Beyond the Introduction It is clear that students knowing only the basic notions would be ill prepared for further studies. Some of the following subjects will be dealt with as time and students' abilities permit.

- Polynomial functions and their special properties, e.g. interpolation, numerical integration, numerical differentiation.
- rational functions
- exponential functions, logarithms and models for growth or decay
- trigonometric functions and elementary models for vibrations, cyclometric functions
- parametrised curves and motions of mass points, some physics using the calculus
- some examples for vector fields and ODEs, population models, linear ODEs with constant coefficients [e.g. as eigenvalue problems], vibrations.

5 Conclusion

The calculus may be automatised. This may be one reason why the calculus may be taught quite sucesssfully by drill and practice, by programming humans, so to speak. Our students deserve more general education than this one.

The discussion, motivation and careful introduction of *basic ideas* and specific notions and their interrelationship is the fabric of analysis. The operations of the calculus may be automated and left to be performed by using a CAS. The meaning of what we are doing by using the calculus with or without an automaton is to be found in studying analysis. Learning analysis must be based on gaining understanding and insight. We have a cultural heritage to pass on to the next generation – not just a finite set of rules. It is fair to answer two questions: What does it mean? – How does it work? While the working may depend on some technology, the meaning is deeply rooted in two millennia of mathematical culture. By using scaffolding and the black-box – white-box principle we may ease the work to be done, and devote more time and efford to well chosen examples which show the meaning of what we teach.

1. *Semantics* is the essence of analysis, a part of mathematics dealing with the continuum, real numbers, functions and limits. For the sake of general education the *Archimedian trick* based on *discretisation* combined with *idealisation* is the center piece. It explains why and how many of the applications of analysis work. Why can the derivative be used to compute instantaneous velocity, instantaneous acceleration, the slope of a tangent, the density of a mass distribution? Why do we need integration to find the total charge from a charge density? Why is there a well defined mean for any continuous function and any compact interval of its domain? Why can we use integration for measuring curve length, areas as well as volumes? Why can we use the derivative to determine the surface of a sphere? Why can the surface of a sphere also be determined by integration? How can we find the centre of mass of a mass distribution in a 'decent' finite domain? What about smooth curves compared to fractal shapes? And why did it take humanity and the best of its thinkers 2000 years or more to develop analysis out of Greek mathematics? Why are there tasks in analysis no computer may solve exactly, and what prevents computers from operating on typical real numbers?

2. *Syntax* is the essence of the calculus which reduces operations in a continuum and rooted in analysis to *finite algebraic operations.* This algorithmic part avoids dealing with infinities and even the computation of limits is passed on to algebraic manipulations thanks to the Bernoulli-Hôpital rule. The syntax of the calculus is essential and is at the heart of many tools of modern science and technology. However it is not expected to convey insight and understanding and by its finitistic nature it cannot reach the fundaments of analysis. Moreover, the computational power available not only allows the construction of reliable CAS software but it is necessary too when numerical methods are to be applied. *Numerical procedures* usually do not follow from the calculus but rather need the insights of analysis and algorithmics to be well designed and applied correctly.

   The calculus is a powerful tool as long as we stay inside the range of *elementary functions.* However, the elementary functions are scarce exceptions in the function spaces dealt with by analysis. Moreover, the elementary functions are closed with respect to the derivative but *not* with respect to primitive functions. An exclusive use of technology bears the risk, to convey the naive belief that all practically relevant functions are elementary.

No engineer would want to turn back to the slide rule or the trigonometric tables. The tools of the 19th century are definitely outdated. We need creative minds developping and using new tools for the present and the future. Calculus will be among these tools, and analysis will remain its basis. However it has to be augmented by numerical and statistical methods based on corresponding software and hardware which enable substantial amounts of data to be exploited. Up to the automatised data collection and data processing, data was scarce. The calculus never showed a big appetite for data. Its central and unrivalled position in the pre-information age may be a consequence of this fact. As soon as we are confronted with massive data, we may notice that data is rarely consistent with simple smooth models we are used to deal with in analysis. Hence it is essential to exploit modern technology to perform the statistical analysis of data and data smoothing. This has become an important first step necessary to prepare the ground for tools of the calculus or methods from analysis to apply.

Let's open the way for a calculus enhanced and combined with other algorithms taking advantage of the treasures of big data. This aim cannot be reached without understanding the basic ideas of our intellectual, technological and material tools and heritage.

The use of a CAS in the teaching of mathematics may save some time necessary to dive into the interesting genesis of ideas leading to analysis. Education that neglects important ideas in favor of drill and practice has to be questioned. We may choose to show big ideas to the next generation with the help of technology like graphing computers, CAS, tablets and more to come. Technology is part of our culture but it will continue to change in contrast to the long lasting and deep ideas in mathematics. We have to take our time to TEACH FUNDAMENTAL IDEAS AND DEMONSTRATE THEIR VALUES ALSO IN APPLICATIONS.

Final Comments and Pointers to the Literature

I have given an account for teaching analysis and the calculus according to the genetic approach and my own experience. In line with my motivation, the focus is on a special approach to teaching analysis at the pre-university level. My message is the combination of the genetic method with the extensive use of technology based on scaffolding and the blackbox–whitebox principles.

I freely admit a somewhat sloppy use of hints to historical facts and a neglect of the spatial dimension. Knowlegde of sources from India, China, Japan would have done justice to history by including extra-european roots of important mathematical ideas too. Moreover, it is plausible to suspect an exchange of germs and genes as well as ideas and knowledge over primarily commercial channels like the silk road.

My impression is that the didactical focus on the genetic method may justify some lacking completeness of the established historical records. The history of the subject, as known today, is dealt with adequately in [ARN], [EDW], [H&W], [STI].

The *genetic method* was propagated first by Otto Toeplitz since 1926. An authoritative source is [TOE], based on Toeplitz's introductory courses at the university level. David Bressoud, in his preface to [TOE], praises the method as an antidote to the undesired early pseudo-formalisation introduced by New Math. Further texts with similar orientation are [POL], [SIM].

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